Mean-field results for the two-component model

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In previous papers we have introduced a new dynamical model of Ising spins: the two-component (TC) model. In this paper we formulate a mean-field version of the TC model by putting it on a complete graph. With such an approach we are able to describe the kinetics in terms of a one-dimensional stochastic process with hopping probabilities depending on the magnetization. This allows us to understand the differences in relaxation between phases observed previously in computer simulations for the TC model on the square lattice

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I. INTRODUCTION

The Ising spin system is undoubtedly one of the most frequently used models of statistical mechanics. Recently, this model has found numerous applications in sociophysics simulations [1,2]. Conversely, as was noticed by Slanina and Lavicka [3], sociologically inspired models pose new challenges to statistical physics.

The two-component (TC) model [4,5] has been proposed to generalize the Sznajd model (for a review see [6]), which was aimed at describing global social phenomena (sociology) by local social interactions (described by social psychology). The crucial difference of the Sznajd model compared to other Ising-type models is that information flows outward.

Because of the importance of this feature, we have decided to introduce a generalized model which kept our old dynamics (the outflow of information), but introduced a function controlling whether a spin should be flipped or not. The model we have presented consists of two components (hence the name TC model) [4].

(i) *Dynamics*. The information flows outward; i.e., a pair of spins S_i and S_{i+1} is chosen to change their nearest neighbors.

(ii) *Disagreement function*. The change of spins is controlled by a certain function [based on anisotropic next-nearest-neighbor Ising (ANNNI) Hamiltonian [7]], which is locally minimized.

In one dimension [4] simulations and analytical reasoning show that the TC model depending on two interaction coefficients can eventually lead the system to one of four phases: (A) degenerated, in which two qualitatively different steady states exist, (B) ferromagnetic, (C) antiphase (2,2), and (D) antiferromagnetic.

In two dimensions [5] the TC model can also eventually lead the system to one of four phases: (A) degenerated, in which four (instead of two) qualitatively different steady states exist, (B) ferromagnetic, (C) double degenerated (instead of no degeneration), and (D) antiferromagnetic.

It has been noted that in our model a degeneration of the steady state was possible even if both coupling constants were greater or smaller than zero, while in the ANNNI [7] model the coupling constants needed to have opposite signs in order to obtain degeneration.

Moreover, it has been shown that the system can relax in two different ways depending on the interaction coefficients [4]. Surprisingly, the system is reaching the final (steady) state in two different ways. In phases (A) and (B) the system behaves "blindly," making a "random" walk to the final state, and in phases (C) and (D) the system is almost ordered after several Monte Carlo steps and then decreasing oscillations around the final state are leading the system into this state. This interesting result has been obtained only by computer simulations.

In this paper we will present a mean-field approach to the TC model. We will solve the model analytically using this approach and will show that there are two kinds of relaxation, depending on interaction coefficients, in agreement with simulation results.

II. TC MODEL

Let us now describe the TC model introduced in [4]. We investigate a system of Ising spins on a one-dimensional lattice. We choose at random a pair of spins S_{i+1} and S_{i+2} and we calculate the disagreement function

$$E^+ = E(S_i, S_{i+1}, S_{i+2}) = -J_1 S_i S_{i+1} - J_2 S_i S_{i+2}. \tag{1}$$

Next we calculate

$$E^- = E(-S_i, S_{i+1}, S_{i+2}) = J_1 S_i S_{i+1} + J_2 S_i S_{i+2} \tag{2}$$

in the case of flipped a ith spin. If $E^- < E^+$, then we flip the ith spin; if not, the spin will remain unchanged. We do the same for the second neighbor of the chosen pair—i.e., for the spin S_{i+3}.

We can easily calculate the disagreement function $E(S_i, S_{i+1}, S_{i+2})$ for all possible triplets: (i) $\uparrow\uparrow\uparrow$, $\downarrow\downarrow\downarrow$, $E_1 \equiv E_1(S_i, S_{i+1}, S_{i+2}) = -(J_1 + J_2)$,

(ii) $\uparrow\uparrow\downarrow$, $\downarrow\downarrow\uparrow$, $E_2 \equiv E_2(S_i, S_{i+1}, S_{i+2}) = -J_1 + J_2$,

(iii) $\uparrow\downarrow\uparrow$, $\downarrow\uparrow\downarrow$, $E_3 \equiv E_3(S_i, S_{i+1}, S_{i+2}) = J_1 - J_2$,

(iv) $\downarrow\uparrow\uparrow$, $\uparrow\downarrow\downarrow$, $E_4 \equiv E_4(S_i, S_{i+1}, S_{i+2}) = J_1 + J_2$.

The definition of the model (change of S_i according to S_{i+1} and S_{i+2}) implies that only two transitions are possible, $E_1 \leftrightarrow E_4$ and $E_2 \leftrightarrow E_3$. This defines four phases:

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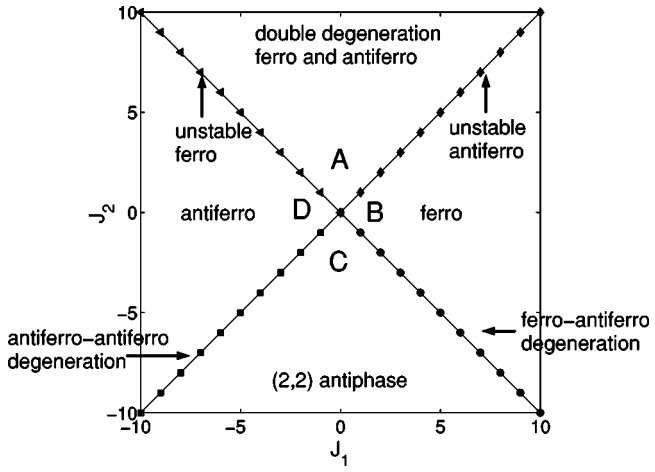

FIG. 1. Phase diagram of the one-dimensional TC model.

(A) $|J_1|<J_2: E_1<E_4, E_3<E_2$,
(B) $|J_2|<J_1: E_1<E_4, E_2<E_3$,
(C) $J_2<|J_1|: E_4<E_1, E_2<E_3$,
(D) $J_1<|J_2|: E_4<E_1, E_3<E_2$.

This means that, e.g., in phase A the ferromagnetic and antiferromagnetic triplets are preferable—i.e., $\downarrow\uparrow\uparrow \rightarrow \uparrow\uparrow\uparrow, \uparrow\uparrow\downarrow \rightarrow \downarrow\uparrow\downarrow$. Thus in one dimension we expect a double degeneration of the steady state: the ferromagnetic and antiferromagnetic steady states should be possible. The full phase diagram of the one-dimensional model consists of four different phases (see Fig. 1):

(A) for $|J_1|<J_2$ there are two possible steady states—ferromagnet and antiferromagnet,

(B) for $J_1>|J_2|$ the final steady state of the system is the ferromagnetic state,

(C) for $|J_1|>J_2$ the final steady state of the system is antiphase (2,2),

(D) for $J_1<|J_2|$ the final steady state of the system is the antiferromagnetic state.

III. RELAXATION IN THE TC MODEL

Recently the following question has been raised by Spirin *et al.* [8,9]: What happens when an Ising ferromagnet, with spins endowed with Glauber dynamics, is suddenly cooled from a high temperature to zero temperature?

The same question can be asked with respect to the TC model. We start from completely random initial conditions and monitor the evolution of the system.

In a previous paper [4] we have found in computer simulations that the relaxation of the system described by the TC model depends strongly on the phase (i.e., interaction coefficients). We have calculated magnetization defined as

$$m(t) = \frac{1}{N}\sum_{i+1}^{N} S_i(t) \tag{3}$$

using Monte Carlo methods and have obtained the following results (see Fig. 2). In phases (A) and (B) the system behaves "blindly," making a "random" walk to the final state, and in phases (C) and (D) the system is almost ordered after several

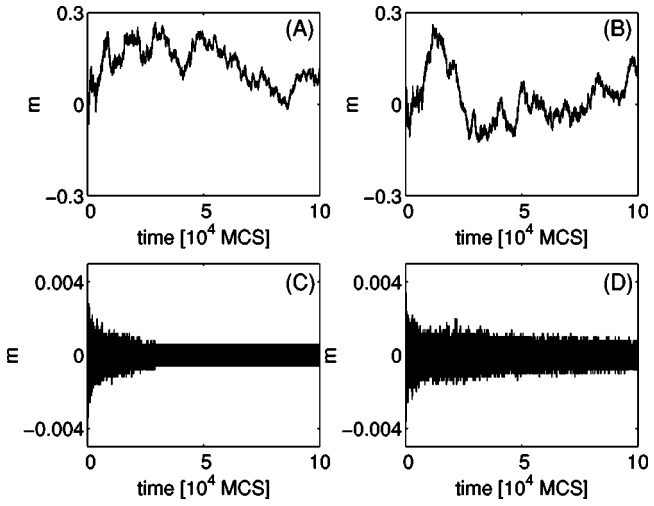

FIG. 2. A sample evolution of the magnetization in four phases (A)–(D) for the TC model on the chain. In phases (A) and (B) the magnetization behaves like a random walker and in phases (C) and (D) the magnetization decreases by damping oscillations.

Monte Carlo steps and then decreasing oscillations around the final state are leading the system into this state. In this paper we want to present a mean-field approach to this problem, analogously as has been done for the Sznajd model [3] and for an irreversible monomer-monomer model of heterogeneous catalysis [10]. It has been proposed to investigate the model on a fully connected network (complete graph), which was, in fact, a mean-field-like approach. In such an approach any two sites are neighbors. Thus in this case we simply choose sites i,j,k at random and change the site k according to sites i and j:

$$E = -J_1 S_k S_j - J_2 S_k S_i. \tag{4}$$

If $\Delta E = -2E < 0$, then we flip the kth spin; otherwise, we leave it unchanged.

We start by presenting computer simulations for the TC model on the complete graph. In Fig. 3 we plot the evolution of magnetization changes for each phase. Again, as for the TC model on the one-dimensional lattice, relaxation depends on the phase. In phases (A) and (B) the magnetization behaves like a random walker, while in phases (C) and (D) it oscillates around zero. However, one should notice that on the complete graph only the ferromagnetic steady state can be reached. This explains the difference between simulation results for the TC model on the chain and on the complete graph for phases (C) and (D). Oscillations are not decreasing for the TC model on the fully connected graph, because the system is not going to reach the final state $m=0$ (which is attainable on the chain).

IV. ANALYTICAL RESULTS

Magnetization m in the system of N spins can change in time due to the two following events:

(i) If $S=-1 \rightarrow S=1$, then the magnetization increases by $2/N$.

(ii) If $S=1 \rightarrow S=-1$, then the magnetization decreases by $2/N$.

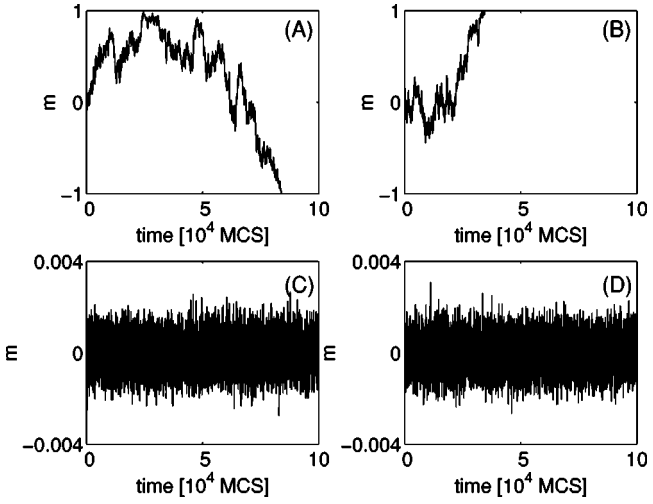

FIG. 3. A sample evolution of the magnetization in four phases (A)–(D) for the TC model on the complete graph. In phases (A) and (B) the magnetization behaves like a random walker and in phases (C) and (D) the magnetization oscillates around zero.

If we denote by N_+ the number of up spins and by N_- the number of down spins, then we can write the magnetization defined by Eq. (3) as

$$m = \frac{N_+ - N_-}{N}. \tag{5}$$

Moreover,

$$N_+ + N_- = N. \tag{6}$$

From these two equations we get

$$N_+ = \frac{N(1+m)}{2} \tag{7}$$

and

$$N_- = \frac{N(1-m)}{2}. \tag{8}$$

In one time step τ, three events are possible: the magnetization increases by $2/N$, decreases by $2/N$, or remains constant:

$$\gamma^+(m) = \mathrm{Prob}\left\{ m \to m + \frac{2}{N} \right\},$$

$$\gamma^-(m) = \mathrm{Prob}\left\{ m \to m - \frac{2}{N} \right\},$$

$$\gamma^0(m) = \mathrm{Prob}\{ m \to m \}. \tag{9}$$

Instead of presenting the discrete master equation, it is simpler and intuitively more revealing to employ a continuum description [10]. We can write the master equation as

$$P_{t+\tau}(m) = \gamma^0(m)P_t(m) + \gamma^+(m-2/N)P_t(m-2/N) + \gamma^-(m+2/N)P_t(m+2/N), \tag{10}$$

where

$$P_{t+\tau}(m) = P_t(m) + \tau \frac{\partial P_t(m)}{\partial t}. \tag{11}$$

Thus,

$$P_t(m) + \tau \frac{\partial P_t(m)}{\partial t} = \gamma^0(m)P_t(m) + \gamma^+(m-2/N)P_t(m-2/N) + \gamma^-(m+2/N)P_t(m+2/N). \tag{12}$$

Expanding $P_t(m-2/N)$, $P_t(m+2/N)$, $\gamma^+(m-2/N)$ and $\gamma^-(m+2/N)$ we get

$$\begin{aligned} P_t(m) + \tau\frac{\partial P_t(m)}{\partial t} &= \gamma^0(m)P_t(m) \\ &+ \left(\gamma^+(m) - \frac{2}{N}\frac{\partial \gamma^+(m)}{\partial m} + \frac{4}{2N^2}\frac{\partial^2 \gamma^+(m)}{\partial m^2} \right) \\ &\times \left(P_t(m) - \frac{2}{N}\frac{\partial P_t(m)}{\partial m} + \frac{4}{2N^2}\frac{\partial^2 P_t(m)}{\partial m^2} \right) \\ &+ \left(\gamma^-(m) + \frac{2}{N}\frac{\partial \gamma^-(m)}{\partial m} + \frac{4}{2N^2}\frac{\partial^2 \gamma^-(m)}{\partial m^2} \right) \\ &\times \left(P_t(m) + \frac{2}{N}\frac{\partial P_t(m)}{\partial m} + \frac{4}{2N^2}\frac{\partial^2 P_t(m)}{\partial m^2} \right). \end{aligned} \tag{13}$$

Because $\gamma^0(m) + \gamma^-(m) + \gamma^+(m) = 1$, we obtain

$$\begin{aligned} \tau\frac{\partial P_t(m)}{\partial t} &= \frac{2}{N}(\gamma^- - \gamma^+)\frac{\partial P_t(m)}{\partial m} + \frac{4}{2N^2}(\gamma^- + \gamma^+)\frac{\partial^2 P_t(m)}{\partial m^2} \\ &+ \frac{2}{N}\left(P_t(m) + \frac{4}{2N^2}\frac{\partial^2 P_t(m)}{\partial m^2} \right)\frac{\partial(\gamma^- - \gamma^+)}{\partial m} \\ &+ \frac{4}{N^2}\frac{\partial P_t(m)}{\partial m}\frac{\partial(\gamma^- + \gamma^+)}{\partial m} + \frac{4}{2N^2}\left(P_t(m) \right. \\ &\left. + \frac{4}{2N^2}\frac{\partial^2 P_t(m)}{\partial m^2} \right)\frac{\partial^2(\gamma^- + \gamma^+)}{\partial m^2} + O\left(\frac{1}{N^3}\right). \end{aligned} \tag{14}$$

In the next sections we calculate the hopping probabilities $\gamma^-(m)$, $\gamma^+(m)$ for each of the four phases, which allows us to find the evolution equations and understand the differences in relaxation between phases.

A. Phase A

In this phase the following transitions are possible: $\downarrow\downarrow\uparrow \to \downarrow\downarrow\downarrow$, $\uparrow\downarrow\downarrow \to \uparrow\downarrow\uparrow$, $\uparrow\uparrow\downarrow \to \uparrow\uparrow\uparrow$, $\downarrow\uparrow\uparrow \to \downarrow\uparrow\downarrow$. We introduce the probabilities p_+, p_- of finding an up spin and a down spin, respectively:

$$N_+ = \frac{N(1+m)}{2}, \quad p_+ = \frac{N_+}{N} = \frac{1+m}{2} \tag{15}$$

and

$$N_- = \frac{N(1-m)}{2}, \quad p_- = \frac{N_-}{N} = \frac{1-m}{2}. \tag{16}$$

If we assume that $N \to \infty$, we get the hopping probabilities

$$\gamma^+(m) = \mathrm{Prob}\left\{ m \to m + \frac{2}{N} \right\} = p_+p_+p_- + p_-p_+p_- = \frac{(1-m^2)}{4},$$

$$\gamma^-(m) = \mathrm{Prob}\left\{ m \to m - \frac{2}{N} \right\} = p_-p_-p_+ + p_+p_-p_- = \frac{(1-m^2)}{4},$$

$$\gamma^0 = \mathrm{Prob}\{m \to m\} = 1 - \frac{1-m^2}{2} = \frac{1+m^2}{2}. \tag{17}$$

As we see,

$$\gamma^+ = \gamma^- = \frac{(1-m^2)}{4}, \tag{18}$$

which is the case of a random walk with steady state for $m^2=1$. This result agrees with simulation results. Now we can write our evolution equation (14). Since in this phase

$$\gamma^- - \gamma^+ = 0 \tag{19}$$

and

$$\gamma^- + \gamma^+ = \frac{1}{2}(1-m^2), \tag{20}$$

we get

$$N^2\tau \frac{\partial P_t(m)}{\partial t} = (1-m^2)\frac{\partial^2 P_t(m)}{\partial m^2} - 4m\frac{\partial P_t(m)}{\partial m} - 2P_t(m). \tag{21}$$

Finally, the dynamical equation is

$$N^2\tau \frac{\partial P_t(m)}{\partial t} = \frac{\partial^2}{\partial m^2}[(1-m^2)P_t(m)]. \tag{22}$$

If we count time in units

$$\tau = \frac{1}{N^2}, \tag{23}$$

we get the evolution equation with only one diffusion term:

$$\frac{\partial P_\tau(m)}{\partial \tau} = \frac{\partial^2}{\partial m^2}[(1-m^2)P_\tau(m)]. \tag{24}$$

Notice that the diffusion coefficient is state dependent, $D(m)=1-m^2$ and $D(m)\to 0$ as $m \to \pm 1$. The evolution of the magnetization therefore can be viewed as the motion of a random walker in a medium that is increasingly 'sticky' near extremities. It is worth mentioning that the magnetization in this phase behaves similarly to the concentration of particles for an irreversible monomer-monomer model of heterogeneous catalysis [10]. Exactly the same behavior of the magnetization was found for the Sznajd model in Ochrombel simplification on the complete graph [3].

The evolution equation of the form (24) has been already studied in Refs. [3,10] and the complete solution of this equation can be found in [3]. One can look for the solution using expansion in eigenvectors and write Eq. (24) in the form

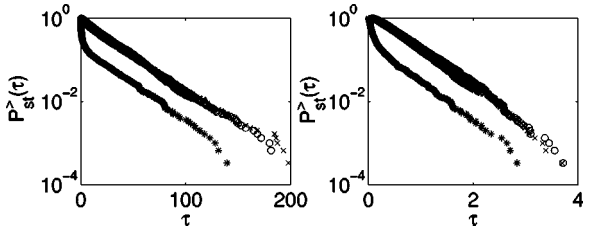

FIG. 4. Probabilities of reaching the stationary state (in phases A and B) in time larger than τ, for the chain (in the left panel) and for the complete graph (in the right panel). The system size $N=10^3$. The values of the initial up-spin concentration $p_+=0.05$ (*), 0.3 ($\times$), and 0.4 ($\bigcirc$). The distribution of waiting times has an exponential tail with the exponent independent of p_+ for both types of networks (the chain and the complete graph), which agrees with the analytical prediction.

$$\frac{\partial}{\partial \tau}P_\tau(m) = LP_\tau(m), \tag{25}$$

where the linear operator L acts as

$$(Lf)(x) = \frac{\partial^2}{\partial x^2}[(1-x^2)f(x)]. \tag{26}$$

The full solution of Eq. (24) can be expanded as

$$P_\tau(m) = \sum_n A_n e^{-n\tau}\Phi_n(m), \tag{27}$$

where $\Phi_n(m)$ are eigenfunctions corresponding to the eigenvalues $-n$ and the coefficients A_n are determined by the initial conditions. The eigenfunctions of the time-dependent eigenvalue equation are the Gegenbauer polynomials [10]. It has been found in [3] that for the δ-initial conditions $P_0(m)=\delta(m-m_0)$ and for $\tau\to\infty$ the probability density

$$P_\tau(m) = \frac{1-m_0}{2}\delta(m+1) + \frac{1+m_0}{2}\delta(m-1). \tag{28}$$

It is easy to see that for an initial magnetization $m_0=2p_+-1$, where p_+ denotes density of up spins, the probability of ending in the state of all +1 is simply p_+.

From the solution (27) we can deduce an important feature for the distribution of waiting times needed to reach the stationary state. Recall that for the δ-initial conditions the distribution of waiting times has an exponential tail [3]:

$$P_s t^{>}(\tau) \approx \frac{6}{4}(1-m_0^2)e^{-2\tau}, \quad \tau \to \infty. \tag{29}$$

Monte Carlo simulations confirm this prediction both on the complete graph and on the chain (see Fig. 4). The distribution of waiting times has an exponential tail with the exponent independent of p_+ for both types of networks (chain and complete graph), which agrees with the analytical prediction. However, relaxation on the chain is much slower than on the complete graph. Moreover, for the chain the exponent grows with the system size, at variance with the results for the complete graph. This is also visible in the mean saturation time, which can be calculated as [3]

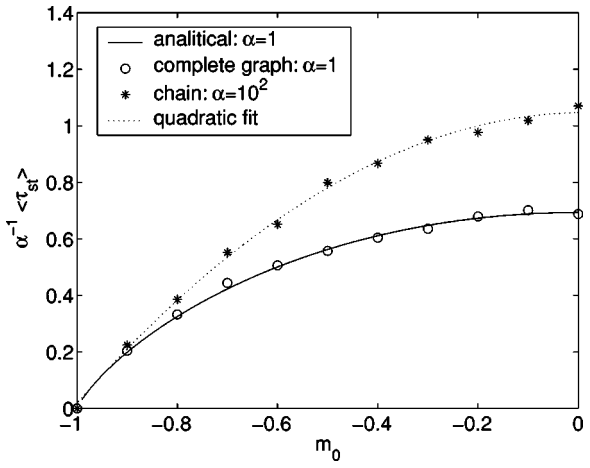

FIG. 5. Mean relaxation time (divided by α, to present results for both networks on the same plot) from the initial state, containing randomly distributed $p_+=(m_0+1)/2$ up spins, to the ferromagnetic steady state in phases A and B. The system size $N=10^3$ and the number of averaging is 3×10^3. It is seen that data for the complete graph are very well fitted by the analytical results (solid line) described by Eq. (30). On the chain the relaxation is much slower. Dotted lines represent a quadratic fit, provided here for illustration purposes only.

$$\langle \tau_{st}(m_0)\rangle = -\frac{m_0}{2}\ln\frac{1+m_0}{1-m_0} - \frac{1}{2}\ln\frac{1-m_0^2}{4}. \tag{30}$$

For the complete graph the saturation time $\langle \tau_{st}(m_0)\rangle$ does not depend on the system size and is very well described by Eq. (30) (see Fig. 5). On the chain the relaxation is much slower and the mean saturation time $\langle \tau_{st}(m_0)\rangle$ grows dramatically with the lattice size. For example, $\langle \tau_{st}(0.5)\rangle \approx 50$ for $N=1000$ and $\langle \tau_{st}(0.5)\rangle \approx 70$ for $N=1500$.

B. Phase B

In this phase the following transitions are possible: $\downarrow\downarrow\uparrow \to \downarrow\downarrow\downarrow$, $\uparrow\downarrow\uparrow \to \uparrow\downarrow\downarrow$, $\uparrow\uparrow\downarrow \to \uparrow\uparrow\uparrow$, $\downarrow\uparrow\downarrow \to \downarrow\uparrow\uparrow$. The hopping probabilities are

$$\gamma^+(m) = \mathrm{Prob}\left\{m \to m+\frac{2}{N}\right\} = p_+p_+p_- + p_-p_+p_- = \frac{(1-m^2)}{4},$$
$$\gamma^-(m) = \mathrm{Prob}\left\{m \to m-\frac{2}{N}\right\} = p_-p_-p_+ + p_+p_-p_+ = \frac{(1-m^2)}{4}. \tag{31}$$

Which is the same result as for phase A.

C. Phase C

In this phase the following transitions are possible: $\downarrow\downarrow\downarrow \to \downarrow\downarrow\uparrow$, $\uparrow\uparrow\uparrow \to \uparrow\uparrow\downarrow$, $\uparrow\downarrow\uparrow \to \uparrow\downarrow\downarrow$, $\downarrow\uparrow\downarrow \to \downarrow\uparrow\uparrow$. The hopping probabilities are

$$\gamma^+(m) = \mathrm{Prob}\left\{m \to m+\frac{2}{N}\right\} = p_-^2(p_+ + p_-) = p_-^2 = \left(\frac{1-m}{2}\right)^2,$$
$$\gamma^-(m) = \mathrm{Prob}\left\{m \to m-\frac{2}{N}\right\} = p_+^2(p_+ + p_-) = p_+^2 = \left(\frac{1+m}{2}\right)^2. \tag{32}$$

Note that in this case the magnetization will oscillate around zero, because

$$\gamma^-(m) - \gamma^+(m) = m: \tag{33}$$

i.e., for $m>0$, $\gamma^-(m)>\gamma^+(m)$ and the magnetization decreases and for $m<0$, $\gamma^-(m)<\gamma^+(m)$ and the magnetization increases. This agrees with the simulation results.

Since in this phase

$$\gamma^-(m) - \gamma^+(m) = m \tag{34}$$

and

$$\gamma^-(m) + \gamma^+(m) = \frac{1+m^2}{2}, \tag{35}$$

the evolution equation (14) has the following form:

$$N\tau\frac{\partial P_t(m)}{\partial t} = \frac{1}{N}\frac{\partial^2}{\partial m^2}[(1-m^2)P_t(m)] + 2\frac{\partial}{\partial m}[mP_t(m)]. \tag{36}$$

For $N\to\infty$ the second, drift term is dominating. Note that in this case we have different time scaling than in cases A and B. In case C we have to put $\tau=1/N$. Finally we get the evolution equation with only the drift term, proportional to the magnetization:

$$\frac{\partial P_\tau(m)}{\partial \tau} = \frac{\partial}{\partial m}[mP_\tau(m)]. \tag{37}$$

It can be easily verified that the solution has the general form

$$P_\tau(m) = m^{-1}f(e^\tau m) \tag{38}$$

for an arbitrary function $f(y)$. The form of the function $f(y)$ is given by the initial conditions. For example, if the initial condition is a δ function, we obtain

$$\begin{aligned} P_0(m) &= \delta(m-m_0) = m^{-1}f(m) \\ &\Downarrow \\ f(m) &= m\delta(m-m_0) \end{aligned} \tag{39}$$

Thus, the density

$$P_\tau(m) = \delta(e^\tau m - m_0) \tag{40}$$

keeps the same form during the evolution; only the location shifts in time.

D. Phase D

In this phase the following transitions are possible: $\uparrow\downarrow\downarrow \to \uparrow\downarrow\uparrow$, $\downarrow\uparrow\uparrow \to \downarrow\uparrow\downarrow$, $\uparrow\uparrow\uparrow \to \uparrow\uparrow\downarrow$, $\downarrow\downarrow\downarrow \to \downarrow\downarrow\uparrow$. The hopping probabilities are

$$\gamma^+ = \mathrm{Prob}\left\{m \to m+\frac{2}{N}\right\} = p_-^2(p_+ + p_-) = p_-^2 = \left(\frac{1-m}{2}\right)^2,$$

$$\gamma^{-} = \mathrm{Prob}\left\{ m \to m - \frac{2}{N} \right\} = p_{+}^{2}(p_{+} + p_{-}) = p_{+}^{2} = \left(\frac{1+m}{2} \right)^{2}, \tag{41}$$

which is the same result as for phase C.

V. CONCLUSIONS

In previous papers we have proposed a new dynamic model of Ising spins [4,5]. We have observed that the model leads to four different phases. Apart from structural differences between phases the difference in relaxation was found by computer simulations [4]. Surprisingly, the system is reaching the final (steady) state in two different ways. In phases (A) and (B) the system behaves "blindly," making a "random" walk to the final state, and in phases (C) and (D) the system is almost ordered after several Monte Carlo steps and then decreasing oscillations around the final state lead the system to this state.

In this paper we have investigated the kinetics of the TC model in the mean-field limit. By modeling the network as a complete graph, one can describe the kinetics in terms of a one-dimensional stochastic process with hopping probabilities dependent on the magnetization. For phases (A) and (B) the hopping probabilities are

$$\gamma_{AB}^{+} = \gamma_{AB}^{-} = \frac{(1-m^2)}{4}, \tag{42}$$

which is the case of a random walk ending at $m = \pm 1$, and for phases (C) and (D),

$$\gamma_{CD}^{+} = \left(\frac{1-m}{2} \right)^{2},$$

$$\gamma_{CD}^{-} = \left(\frac{1+m}{2} \right)^{2}. \tag{43}$$

It is easy to notice that in the second case the magnetization oscillates around zero, because

$$\gamma^{-} - \gamma^{+} = m. \tag{44}$$

These results agree with simulations of the TC model both on the complete graph and the one-dimensional lattice.

Instead of presenting discrete master equations, it is simpler to employ a continuum description. For phases (A) and (B) a evolution equation with only a diffusive term has been obtained. It has been noticed that the diffusion coefficient was state dependent, $D(m) = 1 - m^2$ and $D(m) \to 0$ as $m \to \pm 1$. The evolution of magnetization therefore can be viewed as the motion of a random walker in a medium that is increasingly "sticky" near the extremities. A similar behavior has been found earlier for the concentration of particles for an irreversible monomer-monomer model of the heterogeneous catalysis [10]. In this case the system eventually reaches the steady state of all spins up or all spins down. The mean relaxation time can be found analytically and for the complete graph computer simulations agree very well with the theory. However, relaxation on the chain is much slower and cannot be described by Eq. (30), since the rescaled mean relaxation time is size dependent and it grows dramatically with the chain length N. For phases (C) and (D) a drift term proportional to the magnetization is dominating in the evolution equation. In this case the density $P_{\tau}(m)$ keeps the same form during the evolution for δ-initial conditions. Because on the complete graph there is only one type of ordering, ferromagnetism and this is also the only steady state that the system can reach, the oscillations of the magnetization in phases (C) and (D) are not damped and no steady state is reached, at variance with the case of the chain.

To summarize the results we provide a list of similarities and differences between theory and simulations.

1. Similarities

(i) In phases (A) and (B) the evolution of magnetization can be viewed as the motion of a random walker in a medium that is increasingly "sticky" near the extremities.

(ii) The ferromagnetic state is the final steady state in phases (A) and (B).

(iii) The mean-field results predict that in phases (A) and (B) for the δ-initial conditions the distribution of waiting times has an exponential tail. This is true for the TC model both on the complete graph and on the chain.

(iv) For the complete graph the saturation time $\langle \tau_{st}(m_0) \rangle$ does not depend on the system size, which is in agreement with the mean-field results and is very well described by Eq. (30).

(v) In phases (C) and (D) the magnetization oscillates around zero.

(vi) Mean-field results predict the steady states with magnetization $m = 0$ in phases (C) and (D). This is indeed the steady state in these phases, although it is never reached on the complete graph (i.e., this is an unstable steady state for the complete graph). However, on the chain such a steady state is reached: in phase (C) it has form of antiphase (2,2) and in phase (D) it is antiferromagnetic.

2. Differences

(i) There are two steady states for the TC model on the chain in the phase (A)—ferromagnetic and antiferromagnetic—while in the mean-field approach only the ferromagnetic steady state is possible.

(ii) In phases (A) and (B) the mean relaxation time and the exponent in the distribution of waiting times on the chain is much larger than predicted by the mean-field approach. Moreover, both grow with the system size at variance with analytical results (and with results for the complete graph).

(iii) In the mean-field approach the oscillations of the magnetization in the sectors (C) and (D) are not damped and no steady state is reached, at variance with the case of the chain.

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